A simple pendulum

Finding the governing equations:

In 2D space, with gravity pointing downwards

A pendulum of mass m on a light string of length l in a gravitational field of intensity g, the pendulum makes an angle θ with the vertical.

In this case, the Lagrangian is the difference between kinetic and gravitational potential energies of the pendulum.

$$L = E_{kinetic} - E_{gravitationalpotential} \tag{1}$$

$$E_{kinetic} = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2 \tag{2}$$

$$E_{g.p.} = mgh = -mgl\cos\theta \tag{3}$$

Then:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{4}$$

Applying the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \tag{5}$$

Finding the components:

$$\frac{\partial L}{\partial\theta} = -mgl\sin\theta \tag{6}$$

$$\frac{\partial L}{\partial\dot{\theta}} = mgl^2\dot{\theta} \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mgl^2\ddot{\theta} \tag{8}$$

Solving the Euler-Lagrange equation:

$$mgl^2\ddot{\theta} + mgl\sin\theta = 0 \tag{9}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{10}$$

This governing equation is familiar as the governing equation of a pendulum. When the pendulum is not subject to any external force and the oscillations are small (such that $sin(\theta) \approx \theta$) then the solution is $\theta = A\cos\sqrt{\frac{g}{l}}t$ where A is a constant depending on initial conditions.

Plotting approximate solutions for these equations for a simple pendulum using a script.

Writing a script in python that follows a really basic numeric solution to this governing equation given some initial conditions.

At the initial timestep: θ and $\dot{\theta}$ are given the user, $\ddot{\theta}$ is then calculated from equation 10 .

At all following time steps, where Δt is the legnth of each time step:

$$\theta_t = \theta_{t-\Delta t} + \dot{\theta}_{t-\Delta t}\Delta t + \frac{1}{2}\ddot{\theta}_{t-\Delta t}\Delta t^2 \tag{11}$$

$$\dot{\theta}_t = \dot{\theta}_{t-\Delta t} + \ddot{\theta}_{t-\Delta t}\Delta t \tag{12}$$

Then $\ddot{\theta}$ can be found from equation 10 as in the initial time step.

Performance improvements and more sophisticated methods can certainly be found, but this should be satisfactory.

Including the effects of air resistance

An important part of the model is its decaying behaviour over time, so the affect of air resistance is added as modification to the calculation of the acceleration term at each time step.

At each time step a modification is made to the acceleration of the pendulum. A force of magnitude αv^2 acts upon the mass of the pendulum, creating an acceleration of $\ddot{\theta} = \frac{\alpha(l\dot{\theta})^2}{ml}$, say, where α is a constant. This acceleration is always in the opposite direction to $\dot{\theta}$.

Then the governing equation at each time step becomes:

$$\ddot{\theta} + \frac{\alpha l}{m}|\dot{\theta}|\dot{\theta} + \frac{g}{l}\sin\theta \tag{13}$$

Other parts of finding the numerical solution are unaffected.

A spherical pendulum

The method from the previous section will do well here, avoiding much of the confusion of using a more Newtonian approach. The Euler-Lagrange equation will be applied twice with relative ease.

Finding the governing equations:

A pendulum of mass m on a light string of length l in a gravitational field on intensity g, the pendulum makes an angle θ with the vertical, and lies in a plane that also contains the vertical, this plane is at an angle ψ to the plane that contains the vertical and the mass at its initial position.

The Lagrangian is again the difference between the kinetic energy of the pendulum and the potential energy of the pendulum. Then:

$$x = l \sin\theta \cos\psi \tag{14}$$

$$y = l \sin\theta \sin\psi \tag{15}$$

$$z = l \cos\theta \tag{16}$$

Using these to find the energies in the pendulum:

$$L = E_{kinetic} - E_{gravitationalpotential} \tag{17}$$

Finding the components:

$$E_{kinetic} = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{ml^2}{2}\Big[\big(\dot\theta\cos\theta\cos\psi - \dot\psi\sin\theta\sin\psi\big)^2 + \big(\dot\theta\cos\theta\sin\psi + \dot\psi\sin\theta\cos\psi\big)^2 + \big(-\dot\theta\sin\theta\big)^2\Big]$$
$$= \frac{ml^2}{2}\Big[\dot\theta^2\cos^2\theta\cos^2\psi - 2\dot\theta\dot\psi\sin\theta\cos\theta\sin\psi\cos\psi + \dot\psi^2\sin^2\theta\sin^2\psi$$
$$+ \dot\theta^2\cos^2\theta\sin^2\psi + 2\dot\theta\dot\psi\sin\theta\cos\theta\sin\psi\cos\psi + \dot\psi^2\sin^2\theta\cos^2\psi + \dot\theta^2\sin^2\theta\Big]$$

$$E_{kinetic} = \frac{1}{2}ml^2\big(\dot\theta^2 + \dot\psi^2\sin^2\theta\big) \tag{18}$$

$$E_{g.p.} = -mgl\cos\theta \tag{19}$$

Producing the Lagrangian:

$$L = \frac{1}{2}ml^2\big(\dot\theta^2 + \dot\psi^2\sin^2\theta\big) + mgl\cos\theta \tag{20}$$

Applying the Euler-Lagrange equation in ψ:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi} \tag{21}$$

$$\frac{\partial L}{\partial \psi} = 0 \tag{22}$$

Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2} m l^2 2\dot{\psi}\sin^2\theta\right] = 0 \tag{23}$$

$$\ddot{\psi}\sin\theta + \dot{\psi}\dot{\theta}2\cos\theta = 0 \tag{24}$$

Dividing by $\sin\theta$, then do not allow $\theta = 0$. This can only occur when there is no angular momentum about the base of the pendulum, detect this in the program and warn if this is the case.

$$\ddot{\psi} = -\frac{2\dot{\psi}\dot{\theta}}{\tan\theta} \tag{25}$$

Applying the Euler-Lagrange equation in θ:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{26}$$

$$\frac{\partial L}{\partial \theta} = m l^2 \dot{\psi}^2 \sin\theta\cos\theta - mgl\sin\theta \tag{27}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} \tag{29}$$

Then:

$$m l^2 \ddot{\theta} - m l^2 \dot{\psi}^2 \sin\theta\cos\theta + mgl\sin\theta = 0 \tag{30}$$

$$\ddot{\theta} = \left(\dot{\psi}^2\cos\theta - \frac{g}{l}\right)\sin\theta \tag{31}$$

Including the affects of air resistance in the governing equations:

Including an acceleration term of $-\frac{\alpha(l\dot{\theta})^2}{ml}$ in the $\dot{\theta}$ direction and an acceleration term of $-\frac{\alpha(l\sin\theta\dot{\psi})^2}{ml\sin\theta}$ in the $\dot{\psi}$ direction in each of equations 25 and 31, then implement the numerical integration procedure as with the simple pendulum.

Creating a script to find an approximate numerical solution to these equations

At the initial time step, the user gives the initial conditions of $\theta, \dot{\theta}, \psi, \dot{\psi}$ and other constants. $\ddot{\theta}$ and $\ddot{\psi}$ are then calculated using:

$$\ddot{\psi} = -\frac{2\dot{\psi}\dot{\theta}}{\tan\theta} - \frac{\alpha l \sin\theta |\dot{\psi}| \dot{\psi}}{m} \tag{32}$$

$$\ddot{\theta} = \left(\dot{\psi}^2 \cos\theta - \frac{g}{l}\right) \sin\theta - \frac{\alpha l |\dot{\theta}| \dot{\theta}}{m} \tag{33}$$

At subsequent time steps, where Δt is the length of each time step:

$$\theta_t = \theta_{t-\Delta t} + \dot{\theta}_{t-\Delta t}\Delta t + \frac{1}{2}\ddot{\theta}_{t-\Delta t}\Delta t^2 \tag{34}$$

$$\dot{\theta}_t = \dot{\theta}_{t-\Delta t} + \ddot{\theta}_{t-\Delta t}\Delta t \tag{35}$$

$$\psi_t = \psi_{t-\Delta t} + \dot{\psi}_{t-\Delta t}\Delta t + \frac{1}{2}\ddot{\psi}_{t-\Delta t}\Delta t^2 \tag{36}$$

$$\dot{\psi}_t = \dot{\psi}_{t-\Delta t} + \ddot{\psi}_{t-\Delta t}\Delta t \tag{37}$$

Then $\ddot{\psi}$ and $\ddot{\theta}$ can be found for that time step with equations 32 and 33 and the movement of the pendulum found over time.